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Proca Effect in Kerr–Newman Metric

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The Proca effect of an electric field is studied in curved space. A Kerr–Newman metric with the photon rest mass can be presented by the analytic continuation (Xu, C. M. (. . . .). *General Relativity* and *Modern Cosmology*, Nanking Normal University) in a short range. It yields the correction in the Kerr–Newman space.

KEY WORDS: Kerr–Newman spacetime; Proca equation; analytic continuation.

1. INTRODUCTION

The Proca equation for photon is the natural extension of the Maxwell equation in the electrodynamics to case with the rest mass (Goldhaber and Nieto, 1971; Proca, 1936). The equation was studied also in curved space that an exact solution for an idealized point particle has yet to be found (Kramer *et al*., 1980; Tucker, private Communication). The Einstein–Proca system has been discussed frequently in the articles, for example in Dereli *et al*. (1996), and has been invoked by Tucker and Wang (1997) in connection with dark matter gravitational interactions. Various approximations were utilized in founding solutions of this system in more complicated cases. Numerical solutions were found independently by Obukov and Vlachynsky (2000) and Toussaint (1999). Vuille *et al*. (2002) solved the Einstein–Proca field equations in point charge field by perturbation analysis, and the result agrees with previous numerical solutions in Reissner–Nordstrom (Proca, 1936).

During the last three decades, photon rest mass problem captured special attention of many investigators (Barrow and Burman, 1984; Chernikov *et al*., 1992; Feinberg, 1969; Fischbach *et al*., 1994; Lakes, 1998; Williams *et al*., 1971). More recently, Lakes (1998), and Luo *et al*. (2003) measured the rest mass of photon by means of torsion balance method, respectively. The most new upper limit on photon rest mass is 1.2×10^{-53} kg (Luo *et al*., 2003). In the universe,

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neutron stars and black holes would be expected to have plentiful electric charges. In this paper, we shall study how they affect the space–time metric for a short range if photon has a rest mass. In the researched situation, each neutron star or black hole can be regarded as a point mass of uniform rate rotation with electric charges and the metric for axis of rotation symmetry can be gotten by the analytic continuation (Xu, ...).

The metric of the axis of rotation symmetry was given, in which the rest mass of photon appears in the negative exponent (Vuille *et al.*, 2002). Moreover, an additive term was expressed by an integral in the metric's component (Vuille *et al.*, 2002). In this paper, our given solution contains all components of the metric in Kerr–Newman spacetime with the photon rest mass by the analytic continuation (Xu, ...).

2. SOLUTION OF THE FIELD EQUATIONS

The equation for a particle exhibiting a spin-1 short or intermediate-range field in flat space is Proca equation (Proca, 1936), which in the absence of currents is

$$\partial_a F^{ab} + \mu^2 A^b = 0 \tag{1}$$

where

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \tag{2}$$

The metric has the diagonal from $(c^2, -1, -1, -1)$. The quantity $\mu^{-1} = \frac{\hbar}{(m_\gamma c)}$ is an effective range of the electromagnetic interaction with m_γ as the photon mass.

The Proca stress energy, unlike the Maxwell stress-energy, is not traceless. Einstein's equations read

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \tag{3}$$

It is advantageous to recast the Proca equation in terms of ordinary partial derivatives (Vuille *et al.*, 2002):

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ab}) + \mu^2 A^b = 0 \tag{4}$$

where g is the determinant of the metric, it turns out that the last term on the right in Eq. (3), which distinguishes the standard Proca from Maxwell.

Obviously, the correction in the Reissner–Nordstrom metric by the photon rest mass (Vuille *et al.*, 2002) will be obtained.

$$g_{00} = e^{\nu} = 1 - \frac{2M}{r} + Q^2 \left(\frac{e^{-2\mu r}}{r^2} + \mu \int_r^{\infty} \frac{e^{-2\mu r}}{r^2} dr \right)$$

and

$$g_{11} = -e^{\lambda} = -\left[1 - \frac{2M}{r} + Q^2\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^{\infty}\frac{e^{-2\mu r}}{r^2}dr\right) + \frac{1}{2}q^2\mu^2 e^{-2\mu r}\right]^{-1}$$
$$g_{22} = -r^2$$
$$g_{33} = -r^2\sin^2\theta \tag{5}$$

In the short case $r \ll \mu^{-1}$ then $\mu \cdot r \ll 1$ So

$$g_{11} = -e^{\lambda} = -\left[1 - \frac{2M}{r} + Q^2\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^{\infty}\frac{e^{-2\mu r}}{r^2}dr\right)\right]^{-1}$$

Next, set

$$r' = r,$$
$$\theta' = \theta,$$
$$\varphi' = \varphi,$$
$$du = dt - \frac{dr}{1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi}\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^{\infty}\frac{e^{-2\mu r}}{r^2}dr\right)} \tag{6}$$

Substitute all these into the Eq. (5) result in

$$d\tau^2 = \left[1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi}\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^{\infty}\frac{e^{-2\mu r}}{r^2}dr\right)\right]du^2$$
$$+ 2dr\,du - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{7}$$

So

$$g^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & \left[1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi}\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^{\infty}\frac{e^{-2\mu r}}{r^2}dr\right)\right] & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & (r\sin\theta)^2 \end{bmatrix} \tag{8}$$

Applying four dimensional complex vectors $\lambda^{\mu}_{(a)}$ and $\tau^{\mu}_{(a)}$ to Eq. (8), we get

$$g^{\mu\nu} = \lambda^{\mu}_{(a)}\tau^{(a)\nu} \tag{9}$$

where

$$\lambda^{\mu}_{(a)} = (-l^{\mu}, -n^{\mu}, m^{\mu}, \bar{m}^{\mu})$$
$$\tau^{(a)\nu} = (n^{\nu}, l^{\nu}, \bar{m}^{\nu}, m^{\nu}) \tag{10}$$

and

$$
\begin{aligned}
l^\mu &= \delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2r}}\left(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right) \\
n^\mu &= \delta_0^\mu - \left[\frac{1}{2} - \frac{GM}{r} + \frac{GQ^2}{8\pi}\left(\frac{e^{-2\mu r}}{r^2} + \mu\int_r^\infty \frac{e^{-2\mu r}}{r^2}dr\right)\right]\delta_1^\mu. \qquad (11)
\end{aligned}
$$

Making the r into complex space by the analytic continuation, we can obtain a new frame

$$
\begin{aligned}
l^\mu &= \delta_1^\mu \\
m^\mu &= \frac{1}{\sqrt{2r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2r}}\left(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right) \\
n^\mu &= \delta_0^\mu - \left[\frac{1}{2} - GM\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) + \frac{GQ^2}{8\pi}\left(\frac{e^{-\mu(r+\bar{r})}}{r\bar{r}} + \mu\int_{\frac{r+r}{2}}^\infty \frac{e^{-\mu(r+\bar{r})}}{r\bar{r}}dr\right)\right]\delta_1^\mu. \qquad (12)
\end{aligned}
$$

The $\bar{r}$ is, of course, the conjugate comlex conjugate of the r. Next, set

$$
\begin{aligned}
u' &= u - ia\cos\theta \\
r' &= r + ia\cos\theta \\
\theta' &= \theta \\
\varphi' &= \varphi. \qquad (13)
\end{aligned}
$$

Where a is constant, interpreted as angular momentum of unit mass, we can obtain the infinitesimal line element matrix of transformation, namely $dx'^\mu = a_\nu^\mu dx^\nu$, and have

$$
a_\nu^\mu = \begin{bmatrix} 1 & 0 & ia\sin\theta & 0 \\ 0 & 1 & -ia\sin\theta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (14)
$$

After transformating, the frame (11) can be written as

$$
l'^\mu = \delta_1^\mu
$$

$$m'^{\mu} = \frac{1}{\sqrt{2}(r' - ia\cos\theta)}\left(ia\sin\theta\left(\delta_0^{\mu} - \delta_1^{\mu}\right) + \delta_2^{\mu} + \frac{i}{\sin\theta}\delta_3^{\mu}\right)$$

$$\bar{m}'^{\mu} = \frac{1}{\sqrt{2}(r' + ia\cos\theta)}\left(-ia\sin\theta\left(\delta_0^{\mu} - \delta_1^{\mu}\right) + \delta_2^{\mu} - \frac{i}{\sin\theta}\delta_3^{\mu}\right),$$

$$n'^{\mu} = \delta_0^{\mu} - \left[\frac{1}{2} - \frac{GMr'}{r'^2 + a^2\cos^2\theta} + \frac{GQ^2}{8\pi}\left(\frac{e^{-\mu r'}}{r'^2 + a^2\cos^2\theta} + \mu\int_{r'}^{\infty}\frac{e^{-\mu r'}}{r'^2 + a^2\cos^2\theta}dr'\right)\right]\delta_1^{\mu}. \quad (15)$$

Getting rid of the "′," we have the form

$$g^{\mu\nu} = \begin{bmatrix} \beta(a^2\sin^2\theta) & -\beta(r^2+a^2) & 0 & \beta a \\ -\beta(r^2+a^2) & \beta\left[2GMr - (r^2+a^2) - \frac{GQ^2}{4\pi}e^{-\mu r}\right] + \frac{GQ^2\mu}{4\pi}\int_r^{\infty}\beta e^{-\mu r}\,dr & 0 & -\beta a \\ 0 & 0 & \beta & 0 \\ \beta a & -\beta a & 0 & \beta\sin^{-2}\theta \end{bmatrix} \quad (16)$$

where

$$\beta = (r^2 + a^2\cos^2\theta)^{-1}.$$

Transforming $u', r', \theta', \varphi'$ into t, r, θ, φ by the Eqs. (13) and (6), we have the solution

$$\begin{aligned} d\tau^2 = {} & \left[1 - \frac{2GMr - \frac{GQ^2}{4\pi}e^{-\mu r}}{r^2 + a^2\cos^2\theta} + \frac{GQ^2}{4\pi}\mu\int_r^{\infty}\frac{e^{-\mu r}}{r^2 + a^2\cos^2\theta}dr\right]dt^2 \\ & - \frac{1}{\frac{r^2+a^2}{r^2 + a^2\cos^2\theta} - \frac{2GMr - \frac{GQ^2}{4\pi}e^{-\mu r}}{r^2 + a^2\cos^2\theta} + \frac{GQ^2}{4\pi}\mu\int_r^{\infty}\frac{e^{-\mu r}}{r^2 + a^2\cos^2\theta}dr}dr^2 \\ & - (r^2 + a^2\cos^2\theta)d\theta^2 \\ & - \left[(r^2+a^2)\sin^2\theta + \left(\frac{2GMr - \frac{GQ^2}{4\pi}e^{-\mu r}}{r^2 + a^2\cos^2\theta} - \frac{GQ^2}{4\pi}\right.\right. \\ & \left.\left.\times\ \mu\int_r^{\infty}\frac{e^{-\mu r}}{r^2 + a^2\cos^2\theta}dr\right)a^2\sin^2\theta\right]d\varphi^2 \\ & + 2a\sin^2\theta\left(\frac{2GMr - \frac{GQ^2}{4\pi}e^{-\mu r}}{r^2 + a^2\cos^2\theta} - \frac{GQ^2}{4\pi}\mu\int_r^{\infty}\frac{e^{-\mu r}}{r^2 + a^2\cos^2\theta}dr\right)d\varphi\,dt \end{aligned} \quad (17)$$

In the limit as $\mu \to 0$, corresponding to an infinite range for the vector potential, a Kerr–Newman spacetime is recovered.

3. CONCLUDING REMARKS

In this paper, a metric in Kerr–Newman spacetime by the photon rest mass is given for the case ($r \ll \mu^{-1}$). In the universe, some stars and black holes almost have very strong electric field. The metric in Kerr–Newman spacetime by the photon rest mass will be obtained by the analytic continuation. The positive Proca terms in the above metric suggest the possibility that some of these objects might be devoid of event horizons, in agreement with the earlier numerical solutions of Obukov and Vlachynsky and Toussaint.

Another interesting property of the above solution is that the gravitational field is repulsive when the constants take on suitable values, since as r gets very small the exponential terms will dominate. One is left to speculate whether such repulsive effects could prevent complete catastrophic gravitational collapse.

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